

Lagrangian of Self-dual Gauge Fields in Various Formulations

Wung-Hong Huang
Department of Physics
National Cheng Kung University
Tainan, Taiwan

ABSTRACT

The Lagrangian of self-dual gauge theory in various formulations are reviewed. From these results we see a simple rule and use it to present some new non-covariant Lagrangian based on the decomposition of spacetime into $D = D_1 + D_2 + D_3$. Our prescription could be easily extended to more complex decomposition of spacetime and some more examples are presented therefore. The self-dual property of the new Lagrangian is proved in detail. We also show that the new non-covariant actions give field equations with 6d Lorentz invariance.

*E-mail: whhwung@mail.ncku.edu.tw

1 Introduction

Chiral p-forms, i.e. antisymmetric boson fields with self-dual (p+1)-form field strengths play a central role in supergravity and in string theory, such as D = 6 and type IIB D = 10 supergravity, heterotic strings [1] and M-theory five-branes [2]. In particular, they contribute to the “miraculous” cancelation of the gravitational anomaly in type-IIB supergravity or superstring theory. The first calculation of the gravitational anomaly for chiral p-forms was performed in [3] without using a Lagrangian but just guessing suitable Feynman rules that incorporate the chirality condition.

It is well known that there is a problem in Lagrangian description of chiral bosons, since manifest duality and spacetime covariance do not like to live in harmony with each other in one action, as first seen by Marcus and Schwarz [4]. Historically, the non-manifestly spacetime covariant action for self-dual 0-form was proposed by Floreanini and Jackiw [5], which is then generalized to p-form by Henneaux and Teitelboim [6]. In general the field strength of chiral p-form $A_{1\dots p}$ is split into electric density $\mathcal{E}^{i_1\dots i_{p+1}}$ and magnetic density $\mathcal{B}^{i_1\dots i_{p+1}}$:

$$\mathcal{E}_{i_1\dots i_{p+1}} \equiv F_{i_1\dots i_{p+1}} \equiv \partial_{[i_1} A_{i_2\dots i_{p+1}]} \quad (1.1)$$

$$\mathcal{B}^{i_1\dots i_{p+1}} \equiv \frac{1}{(p+1)!} \epsilon^{i_1\dots i_{2p+2}} F_{i_{p+2}\dots i_{2p+2}} \equiv \tilde{F}^{i_1\dots i_{p+1}} \quad (1.2)$$

in which \tilde{F} is the dual form of F . The Lagrangian is described by

$$L = \frac{1}{p!} \vec{\mathcal{B}} \cdot (\vec{\mathcal{E}} - \vec{\mathcal{B}}) = \frac{1}{p!} \tilde{F}_{i_1\dots i_{p+1}} (F^{i_1\dots i_{p+1}} - \tilde{F}^{i_1\dots i_{p+1}}) \quad (1.3)$$

Note that in order for self-dual fields to exist, i.e. $\tilde{F} = F$, the field strength F and dual field strength \tilde{F} should have the same number of component. As the double dual on field strength shall give the original field strength the spacetime dimension have to be 2 modulo 4. Above actions, however, lead to second class constraints and complicates the quantization procedure.

Siegel in [7] proposed a manifestly spacetime covariant action of chiral p-form models by squaring the second-class constraints and introducing Lagrange multipliers λ_{ab} into the action. The Lagrangian of chiral 2 form is described by

$$L_{Siegel} = -\frac{1}{12} F_{abc} F^{abc} + \frac{1}{4} \lambda_{ab} \mathcal{F}^{acd} \mathcal{F}^b_{cd} \quad (1.4)$$

in which we define

$$\mathcal{F} \equiv F - \tilde{F} \quad (1.5)$$

It is easy to see that the field equation $0 = \frac{\delta S}{\delta \lambda_{ab}}$ implies $\mathcal{F} = 0$ and we get the self-dual property. Using this property the other field equation $0 = \frac{\delta S}{\delta A_{ab}}$ is automatically satisfied. Siegel action, however, does not have enough local symmetry to completely gauge the Lagrange multipliers away and suffers from anomaly of gauge symmetry.

Note that, the self-dual relation $\tilde{F} = F$ is a first-order differential equation which defines the dynamics of the chiral boson, contrast to other bosonic fields whose equations of motion are usually second-order differential equations. This lead McClain, Wu and Yu to construct chiral field action in a first order form [8]. In this case, for the Lagrange multiplier itself not to carry propagating degrees of freedom one has to introduce an infinite number of auxiliary fields “compensating” the dynamics of each other. However, this infinite set corresponds to the infinite number of local symmetries which cause problems in choosing the right regularization procedure during the quantization.

Pasti, Sorokin and Tonin in 1995 constructed a Lorentz covariant formulation of chiral p-forms in $D = 2(p+1)$ dimensions that contains a finite number of auxiliary fields in a non-polynomial way [9]. For example, 6D PST Lagrangian is

$$L_{PST} = -\frac{1}{6}F_{abc}F^{abc} + \frac{1}{(\partial_q a \partial^q a)}\partial^m a(x)\mathcal{F}_{mnl}\mathcal{F}^{nlr}\partial_r a(x) \quad (1.6)$$

in which $a(x)$ is the auxiliary field. In the gauge $\partial_r a = \delta_r^1$ the PST formulation reduces to the non-manifestly covariant formulation [5,6]. On the other hand, Perry and Schwarz [10] had shown that the non-covariant action (1.3) gives field equations with 6d Lorentz invariance.

Recently, a new non-covariant Lagrangian formulation of a chiral 2-form gauge field in 6D, called as (3+3) decomposition, was derived in [11] from the Bagger-Lambert-Gustavsson (BLG) model [12]. The covariant formulation of the associated Lagrangian is constructed in [13], with the use of a triplet of auxiliary scalar fields. Later, a general non-covariant Lagrangian formulation of self-dual gauge theories in diverse dimensions was constructed [14]. In this general formulation the (2+4) decomposition of Lagrangian was found.

In section 2 we review above formulations of self-dual 2-form in the decomposition of $D = D_1 + D_2$ and find a simple rule. In section 3 we use the rule to construct new non-covariant actions of self-dual 2-form gauge theory in the decomposition of $D = D_1 + D_2 + D_3$. We present a detailed proof about the self-dual property in the new Lagrangian. We also show in detail that the new non-covariant action gives field equations with 6d Lorentz invariance. In section 4 we generalize our prescription to construct a non-covariant action in the decomposition of $D = D_1 + D_2 + D_3 + D_4$. Last section is devoted to a short conclusion.

2 Lagrangian in Decomposition: $D = D_1 + D_2$

To begin with, let us first define a useful function L_{ijk} :

$$L_{ijk} \equiv \tilde{F}_{ijk} \times (F^{ijk} - \tilde{F}^{ijk}), \text{ without summation over indices } i, j, k \quad (2.1)$$

which is useful in the following formulations.

2.1 D=1+5

In the (1+5) decomposition the spacetime index $A = (1, \dots, 6)$ is decomposed as $A = (1, \dot{a})$, with $\dot{a} = (2, \dots, 6)$. Then $L_{ABC} = (L_{1\dot{a}\dot{b}}, L_{\dot{a}\dot{b}\dot{c}})$. In terms of L_{ABC} , the Lagrangian is expressed as [14]

$$L_{1+5} = -\frac{1}{4} \sum L_{1\dot{a}\dot{b}} = -\frac{1}{4} \tilde{F}_{1\dot{a}\dot{b}} (F^{1\dot{a}\dot{b}} - \tilde{F}^{1\dot{a}\dot{b}}), \text{ has summation over } \dot{a} \dot{b} \quad (2.2)$$

In table 1 we show all possible form in L_{ABC} and see that L_{1+5} picks up only $L_{1\dot{a}\dot{b}}$. Self-dual property of L_{1+5} had been proved in [14].

Table 1: Lagrangian in various decompositions: $D = D_1 + D_2$.

| D=1+5 | | | | D=3+3 | | | | D=2+4 | | | |
|-------|-----------------------|-----|--------------------------------------|-------|--------------------------------|-----|--------------------------------------|-------|-----------------------|-----|--------------------------------------|
| 123 | $L_{1\dot{a}\dot{b}}$ | 456 | $\cancel{L_{\dot{a}\dot{b}\dot{c}}}$ | 123 | L_{abc} | 456 | $\cancel{L_{\dot{a}\dot{b}\dot{c}}}$ | 123 | $L_{ab\dot{a}}$ | 456 | $\cancel{L_{\dot{a}\dot{b}\dot{c}}}$ |
| 124 | | 356 | | 124 | $L_{ab\dot{a}}$ | 356 | $\cancel{L_{\dot{a}\dot{b}\dot{c}}}$ | 124 | | 356 | |
| 125 | | 346 | | 125 | | 346 | | 125 | | 346 | |
| 126 | | 345 | | 126 | | 345 | | 126 | | 345 | |
| 134 | | 256 | | 134 | | 256 | | 134 | $L_{a\dot{a}\dot{b}}$ | 256 | $L_{a\dot{a}\dot{b}}$ |
| 135 | | 246 | | 135 | | 246 | | 135 | | 246 | |
| 136 | | 245 | | 136 | | 245 | | 136 | | 245 | |
| 145 | | 236 | | 145 | $\cancel{L_{a\dot{a}\dot{b}}}$ | 236 | $L_{ab\dot{a}}$ | 145 | | 236 | |
| 146 | | 235 | | 146 | | 235 | | 146 | | 235 | |
| 156 | | 234 | | 156 | | 234 | | 156 | | 234 | |

2.2 D=3+3

In the (3+3) decomposition [14] the spacetime index A is decomposed as $A = (a, \dot{a})$, with $a = (1, 2, 3)$ and $\dot{a} = (4, 5, 6)$. Then $L_{ABC} = (L_{abc}, L_{ab\dot{a}}, L_{a\dot{a}\dot{b}}, L_{\dot{a}\dot{b}\dot{c}})$. Using table 1

it is easy to see that in terms of L_{ABC} the Lagrangian can be expressed as [14]

$$L_{3+3} = -\frac{1}{12} \left(\sum L_{abc} + 3 \sum L_{ab\dot{a}} \right) \quad (2.3)$$

Let us make following interesting comments:

1. Why there is the “3” factor before $L_{ab\dot{a}}$ in above equation ? This is because that we have to include three kinds of L_{ijk} : $L_{ab\dot{a}}$, $L_{a\dot{a}b}$ and $L_{\dot{a}ab}$.
2. Note that choosing $L_{3+3} \sim \sum L_{abc} + 3 \sum L_{a\dot{a}b}$ will spoil the gauge symmetry $\delta A_{ab} = \Phi_{ab}$ which is crucial in proving the self-dual property of the Lagrangian. A simple rule to have this symmetry is that the choosing Lagrangian L_{3+3} shall contain all possible index “ab” in L_{ABC} . More precisely, as $L_{ABC} = (L_{abc}, L_{ab\dot{a}}, L_{a\dot{a}b}, L_{\dot{a}bc})$ the all possible term with index “ab” in L_{ABC} is $L_{abc}, L_{ab\dot{a}}$. As both terms have been included in L_{3+3} , the Lagrangian thus has the crucial gauge symmetry. Self-dual property of L_{3+3} had been proved in [13,14].

2.3 D=2+4

In the (2+4) decomposition the spacetime index A is decomposed as $A = (a, \dot{a})$, with $a = (1, 2)$ and $\dot{a} = (3, \dots, 6)$. Then $L_{ABC} = (L_{ab\dot{a}}, L_{a\dot{a}b}, L_{\dot{a}bc})$. From table 1 it is easy to see that in terms of L_{ABC} the Lagrangian can be expressed as [14]

$$L_{2+4} = -\frac{1}{4} \left(\sum L_{ab\dot{a}} + \frac{1}{2} \sum L_{a\dot{a}b} \right) \quad (2.4)$$

Self-dual property of L_{2+4} had been proved in [14]. Let us make following interesting comments:

1. Why there is the $\frac{1}{2}$ factor before $L_{a\dot{a}b}$ in above equation ? This is because that in table 1 $L_{a\dot{a}b}$ contains both of left-line element and right-line element (for example, it includes L_{134} and L_{256}), thus there is double counting.
2. From table 1 we see that the difference between the Lagrangian in decomposition $D = 2+4$ and $D = 1+5$ is that we have chosen left-hand (electric) part and right-hand (magnetic) part in $D = 2+4$, while in $D = 1+5$ we choose only left-hand (electric) part. In the self-dual theory the electric part is equal to magnetic part. Thus the Lagrangian choosing electric part is equivalent to that choosing magnetic part. However, in the decomposition into different direct-product of spacetime one shall choose different part of L_{ijk} to mixing to each other. This renders the results to be different and we have many kinds of formulation, as shown in the next section.

3 Lagrangian in Decomposition: $D = D_1 + D_2 + D_3$

We now consider another decomposition of Lagrangian by: $D = D_1 + D_2 + D_3$

3.1 D=1+1+4

In the (1+1+4) decomposition the spacetime index A is decomposed as $A = (1, 2, \dot{a})$, with $\dot{a} = (3, 4, 5, 6)$, and $L_{ABC} = (L_{12\dot{a}}, L_{\dot{a}\dot{b}\dot{c}}, L_{1\dot{a}\dot{b}}, L_{2\dot{a}\dot{b}})$.

Table 2: Lagrangian in various decompositions: $D = D_1 + D_2 + D_3$.

| D=1+1+4 | | | | D=1+2+3 | | | | D=2+2+2 | | | |
|---------|--------------------------------|-----|---------------------------------------|---------|-----------------------|-----|---------------------------------|---------|-----------------------------------|-----|--|
| 123 | $L_{12\dot{a}}$ | 456 | $\mathcal{L}_{\dot{a}\dot{b}\dot{c}}$ | 123 | \mathcal{L}_{1ab} | 456 | $L_{\dot{a}\dot{b}\dot{c}}$ | 123 | $L_{ab\dot{a}}$ | 456 | $\mathcal{L}_{\dot{a}\dot{a}\dot{b}}$ |
| 124 | | 356 | | 124 | $L_{1a\dot{a}}$ | 356 | $\mathcal{L}_{a\dot{a}\dot{b}}$ | 124 | $L_{ab\ddot{a}}$ | 356 | $\mathcal{L}_{\dot{a}\dot{b}\ddot{a}}$ |
| 125 | | 346 | | 125 | | 346 | | 125 | | 346 | |
| 126 | | 345 | | 126 | | 345 | | 126 | | 345 | |
| 134 | $\alpha_1 L_{1\dot{a}\dot{b}}$ | 256 | $\alpha_2 L_{2\dot{a}\dot{b}}$ | 134 | | 256 | | 134 | $L_{a\dot{a}\dot{b}}$ | 256 | $\mathcal{L}_{a\dot{a}\dot{b}}$ |
| 135 | | 246 | | 135 | | 246 | | 135 | $L_{a\ddot{a}\ddot{a}}$ | 246 | $L_{a\ddot{a}\ddot{a}}$ |
| 136 | | 245 | | 136 | | 245 | | 136 | | 245 | |
| 145 | | 236 | | 145 | $L_{1\dot{a}\dot{b}}$ | 236 | $\mathcal{L}_{ab\dot{a}}$ | 145 | | 236 | |
| 146 | | 235 | | 146 | | 235 | | 146 | | 235 | |
| 156 | | 234 | | 156 | | 234 | | 156 | $\mathcal{L}_{a\ddot{a}\ddot{b}}$ | 234 | $L_{a\dot{a}\dot{b}}$ |

From table 2 it is easy to see that, in terms of L_{ABC} , the Lagrangian can be expressed as

$$L_{1+1+4} = 6 \sum L_{12\dot{a}} + \frac{3(1-\alpha)}{2} \sum L_{1\dot{a}\dot{b}} + \frac{3(1+\alpha)}{2} \sum L_{2\dot{a}\dot{b}} \quad (3.1)$$

We neglect overall constant in Lagrangian, which is irrelevant to the following proof. Note that the case of $\alpha = 0$ is just L_{2+4} , the case of $\alpha = -1$ is just L_{1+5} , and the case of $\alpha = 1$ is just L_{1+5} while exchanging indices 1 and 2, as can be seen from table 1.

We now follow the method in [14] to prove the self-dual property of L_{1+1+4} and follow the method in [10] to prove that the new non-covariant action gives field equations with 6d Lorentz invariance.

3.1.1 Self-duality in D=1+1+4

First, we rewrite the Lagrangian as

$$\begin{aligned}
L_{1+1+4} &= 6\tilde{F}_{12\dot{a}}(F^{12\dot{a}} - \tilde{F}^{12\dot{a}}) + \frac{3(1-\alpha)}{2}\tilde{F}_{1\dot{a}\dot{b}}(F^{1\dot{a}\dot{b}} - \tilde{F}^{1\dot{a}\dot{b}}) \\
&\quad + \frac{3(1+\alpha)}{2}\tilde{F}_{2\dot{a}\dot{b}}(F^{2\dot{a}\dot{b}} - \tilde{F}^{2\dot{a}\dot{b}}) \\
&= -\tilde{F}_{\dot{a}\dot{b}\dot{c}}F^{\dot{a}\dot{b}\dot{c}} + F_{\dot{a}\dot{b}\dot{c}}\tilde{F}^{\dot{a}\dot{b}\dot{c}} - \frac{3(1-\alpha)}{2}\tilde{F}_{2\dot{a}\dot{b}}F^{2\dot{a}\dot{b}} + \frac{3(1-\alpha)}{2}F_{2\dot{a}\dot{b}}\tilde{F}^{2\dot{a}\dot{b}} \\
&\quad + \frac{3(1+\alpha)}{2}\tilde{F}_{1\dot{a}\dot{b}}F^{1\dot{a}\dot{b}} + \frac{3(1+\alpha)}{2}F_{1\dot{a}\dot{b}}\tilde{F}^{1\dot{a}\dot{b}} \quad (3.2)
\end{aligned}$$

The variation of the action S_{1+1+4} gives

$$\frac{\delta S_{1+1+4}}{\delta A_{12}} = -6\partial_{\dot{a}}\tilde{F}^{12\dot{a}} = 0 \quad (3.3)$$

which is identically zero. This means that terms involved A_{12} only through total derivative terms and we have gauge symmetry

$$\delta A_{12} = \Phi_{12} \quad (3.4)$$

for arbitrary functions Φ_{12} . The **Gauge symmetry** is crucial to prove the self-duality in following.

Next, the field equations

$$0 = \frac{\delta S_{1+1+4}}{\delta A_{1\dot{a}}} = -6(\partial_2\tilde{F}^{21\dot{a}} + \partial_{\dot{b}}\tilde{F}^{\dot{b}1\dot{a}}) - 6(1+\alpha)\partial_{\dot{b}}\mathcal{F}^{\dot{b}1\dot{a}} = -6(1+\alpha)\partial_{\dot{b}}\mathcal{F}^{\dot{b}1\dot{a}} \quad (3.5)$$

$$0 = \frac{\delta S_{1+1+4}}{\delta A_{2\dot{a}}} = -6(\partial_1\tilde{F}^{12\dot{a}} + \partial_{\dot{b}}\tilde{F}^{\dot{b}2\dot{a}}) - 6(1-\alpha)\partial_{\dot{b}}\mathcal{F}^{\dot{b}2\dot{a}} = -6(1-\alpha)\partial_{\dot{b}}\mathcal{F}^{\dot{b}2\dot{a}} \quad (3.6)$$

has solutions

$$\mathcal{F}_{1\dot{a}\dot{b}} = \epsilon_{12\dot{a}\dot{b}\dot{c}\dot{d}}\partial^{\dot{c}}\Phi^{2\dot{d}} \quad (3.7)$$

$$\mathcal{F}_{2\dot{a}\dot{b}} = \epsilon_{12\dot{a}\dot{b}\dot{c}\dot{d}}\partial^{\dot{c}}\Psi^{1\dot{d}} \quad (3.8)$$

for arbitrary functions $\Phi^{2\dot{d}}$ and $\Psi^{1\dot{d}}$.

We can now follow [14] to find a self-dual relation. First, taking the Hodge-dual of $\mathcal{F}_{1\dot{a}\dot{b}}$ in above solution and identifying it to the solution $\mathcal{F}_{2\dot{a}\dot{b}}$ in above equation we find that

$$\partial_{\dot{a}}\Phi_{2\dot{b}} = \epsilon_{12\dot{a}\dot{b}\dot{c}\dot{d}}\partial^{\dot{c}}\Psi^{1\dot{d}} \quad (3.9)$$

Acting $\partial^{\dot{a}}$ on both sides gives

$$\partial^{\dot{a}}\partial_{\dot{a}}\Phi^{2\dot{b}} = 0 \quad (3.10)$$

Following [14], imposing the boundary condition that the regular field $\Phi^{2\dot{b}}$ be vanished at infinities will lead to the unique solution $\Phi^{2\dot{b}} = 0$ and we arrive at the self-duality conditions

$$\mathcal{F}_{1\dot{a}\dot{b}} = 0 \quad (3.11)$$

Taking the Hodge-dual of $\mathcal{F}_{1\dot{a}\dot{b}}$ we also obtain

$$\mathcal{F}_{2\dot{a}\dot{b}} = 0 \quad (3.12)$$

Using above result the another field equation becomes

$$\begin{aligned} 0 = \frac{\delta S_{1+1+4}}{\delta A_{\dot{a}\dot{b}}} &= -3(\partial_2 \tilde{F}^{2\dot{a}\dot{b}} + \partial_{\dot{c}} \tilde{F}^{\dot{c}\dot{a}\dot{b}} + \partial_1 \tilde{F}^{1\dot{a}\dot{b}}) \\ &\quad -6\partial_{\dot{c}} \mathcal{F}^{\dot{c}\dot{a}\dot{b}} - 3(1+\alpha)\partial_1 \mathcal{F}^{1\dot{a}\dot{b}} - 3(1-\alpha)\partial_2 \mathcal{F}^{2\dot{a}\dot{b}} \\ &= -6\partial_{\dot{c}} \mathcal{F}^{\dot{c}\dot{a}\dot{b}} \end{aligned} \quad (3.13)$$

which has solution

$$\mathcal{F}_{\dot{a}\dot{b}\dot{c}} = \epsilon_{12\dot{a}\dot{b}\dot{c}d} \partial^d \Phi^{12} \quad (3.14)$$

We can now use the gauge symmetry of $\delta A_{12} = \Phi_{12}$ to totally remove Φ^{12} in $\mathcal{F}_{\dot{a}\dot{b}\dot{c}}$ and we find a self-dual relation

$$\mathcal{F}_{\dot{a}\dot{b}\dot{c}} = 0 \quad (3.15)$$

These complete the proof.

3.1.2 Lorentz Invariance in D=1+1+4

As covariant symmetry on 4D coordinates $x_{\dot{a}}$ is manifest we only need to examine transformations (I) mixing x_1 with $x_{\dot{a}}$, (II) mixing x_2 with $x_{\dot{a}}$ and (III) mixing x_1 with x_2 .

(I) For the mixing x_1 with $x_{\dot{a}}$ we shall consider the transformation

$$\delta x^{\dot{a}} = \omega^{\dot{a}1} x_1 \equiv \Lambda^{\dot{a}} x_1, \quad (3.16)$$

$$\delta x^1 = \omega^{1\dot{a}} x_{\dot{a}} = -\Lambda^{\dot{a}} x_{\dot{a}} = -\Lambda \cdot x \quad (3.17)$$

Define

$$\Lambda \cdot L \equiv (\Lambda \cdot x) \partial_1 - x_1 (\Lambda \cdot \partial) \quad (3.18)$$

then (detailed in Appendix A)

$$\delta F_{12\dot{a}} = (\Lambda \cdot L) F_{12\dot{a}} + \Lambda^{\dot{b}} F_{\dot{b}2\dot{a}} \quad (3.19)$$

$$\delta F_{\dot{a}\dot{b}\dot{c}} = (\Lambda \cdot L) F_{\dot{a}\dot{b}\dot{c}} - \Lambda_{\dot{a}} F_{1\dot{b}\dot{c}} - \Lambda_{\dot{b}} F_{\dot{a}1\dot{c}} - \Lambda_{\dot{c}} F_{\dot{a}\dot{b}1} \quad (3.20)$$

$$\delta F_{1\dot{a}\dot{b}} = (\Lambda \cdot L) F_{1\dot{a}\dot{b}} + \Lambda^{\dot{c}} F_{\dot{c}\dot{a}\dot{b}} \quad (3.21)$$

$$\delta F_{2\dot{a}\dot{b}} = (\Lambda \cdot L) F_{2\dot{a}\dot{b}} - \Lambda_{\dot{a}} F_{21\dot{b}} - \Lambda_{\dot{b}} F_{2\dot{a}1} \quad (3.22)$$

Use above transformation we can find

$$\begin{aligned} \delta \tilde{F}_{12\dot{a}} &= (\Lambda \cdot L) \tilde{F}_{12\dot{a}} + \frac{1}{6} \epsilon_{12\dot{a}\dot{b}\dot{c}\dot{d}} (\delta_{spin} F^{\dot{b}\dot{c}\dot{d}}) \\ &= (\Lambda \cdot L) \tilde{F}_{12\dot{a}} + \frac{1}{6} \epsilon_{12\dot{a}\dot{b}\dot{c}\dot{d}} [-\Lambda^{\dot{b}} F^{1\dot{c}\dot{d}} - \Lambda^{\dot{c}} F^{\dot{b}1\dot{d}} - \Lambda^{\dot{d}} F^{\dot{b}\dot{c}1}] \\ &= (\Lambda \cdot L) \tilde{F}_{12\dot{a}} + \Lambda^{\dot{b}} \tilde{F}_{\dot{a}\dot{b}2} \end{aligned} \quad (3.23)$$

$$\begin{aligned} \delta \tilde{F}_{1\dot{a}\dot{b}} &= (\Lambda \cdot L) \tilde{F}_{1\dot{a}\dot{b}} + \frac{1}{6} \epsilon_{1\dot{a}\dot{b}2\dot{c}\dot{d}} (\delta_{spin} F^{2\dot{c}\dot{d}} \cdot 3) \\ &= (\Lambda \cdot L) \tilde{F}_{1\dot{a}\dot{b}} + \frac{1}{2} \epsilon_{1\dot{a}\dot{b}2\dot{c}\dot{d}} [-\Lambda^{\dot{c}} F^{21\dot{d}} - \Lambda^{\dot{d}} F^{2\dot{c}1}] \\ &= (\Lambda \cdot L) \tilde{F}_{1\dot{a}\dot{b}} + \Lambda^{\dot{c}} \tilde{F}_{\dot{c}\dot{a}\dot{b}} \end{aligned} \quad (3.24)$$

in which $\delta_{spin} F$ is defined in Appendix A. Therefore

$$\delta(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) = (\Lambda \cdot L)(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) + \Lambda^{\dot{b}}(F_{\dot{a}\dot{b}2} - \tilde{F}_{\dot{a}\dot{b}2}) \quad (3.25)$$

$$\delta(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) = (\Lambda \cdot L)(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) + \Lambda^{\dot{c}}(F_{\dot{c}\dot{a}\dot{b}} - \tilde{F}_{\dot{c}\dot{a}\dot{b}}) \quad (3.26)$$

which are zero for self-dual theory and the non-covariant action gives field equations with 6d Lorentz transformation mixing x_1 with $x_{\dot{a}}$.

(II) With exchange the index $1 \leftrightarrow 2$ above result also shows that the non-covariant action gives field equations with 6d Lorentz transformation mixing x_2 with $x_{\dot{a}}$.

(III) Finally, we consider the mixing x_1 with x_2 . The transformation is

$$\delta x^1 = \omega^{12} x_2 \equiv \Lambda x_2, \quad (3.27)$$

$$\delta x^2 = \omega^{21} x_1 = -\Lambda x_1 \quad (3.28)$$

Define

$$\Lambda \cdot L \equiv (\Lambda x_2) \partial_1 - x_1 (\Lambda \partial_2) \quad (3.29)$$

then

$$\delta F_{12\dot{a}} = (\Lambda \cdot L) F_{12\dot{a}} \quad (3.30)$$

$$\delta F_{\dot{a}\dot{b}\dot{c}} = (\Lambda \cdot L) F_{\dot{a}\dot{b}\dot{c}} \quad (3.31)$$

$$\delta F_{1\dot{a}\dot{b}} = (\Lambda \cdot L) F_{1\dot{a}\dot{b}} - \Lambda F_{\dot{a}\dot{b}2} \quad (3.32)$$

$$\delta F_{2\dot{a}\dot{b}} = (\Lambda \cdot L) F_{2\dot{a}\dot{b}} + \Lambda F_{\dot{a}\dot{b}1} \quad (3.33)$$

Use above transformation we can calculate the transformations of $\tilde{F}_{12\dot{a}}$ and $\tilde{F}_{1\dot{a}\dot{b}}$. Then we see that

$$\delta(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) = (\Lambda \cdot L)(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) \quad (3.34)$$

$$\delta(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) = (\Lambda \cdot L)(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) - \Lambda(F_{\dot{a}\dot{b}2} - \tilde{F}_{\dot{a}\dot{b}2}) \quad (3.35)$$

which are zero for self-dual theory and the non-covariant action gives field equations with 6d Lorentz transformation mixing x_1 with x_2 .

In summary, we have found the non-covariant action of self-dual 2-form in decomposition $D = 1 + 1 + 4$ and have checked that the non-covariant action gives field equations with 6d Lorentz transformation.

3.2 D=1+2+3

In the (1+2+3) decomposition the spacetime index A is decomposed as $A = (1, a, \dot{a})$, with $a = (2, 3)$, $\dot{a} = (4, 5, 6)$, and $L_{ABC} = (L_{1ab}, L_{1a\dot{a}}, L_{1\dot{a}\dot{b}}, L_{\dot{a}\dot{b}\dot{c}}, L_{a\dot{a}\dot{b}}, L_{ab\dot{a}})$. From table 2 it is easy to see that, in terms of L_{ABC} , the Lagrangian can be expressed as

$$L_{1+2+3} = \sum L_{\dot{a}\dot{b}\dot{c}} + 6 \sum L_{1a\dot{a}} + 3 \sum L_{1\dot{a}\dot{b}} \quad (3.36)$$

Choosing $L_{1ab} + L_{1a\dot{a}} + L_{1\dot{a}\dot{b}}$ is just L_{1+5} , and choosing $L_{1ab} + L_{1a\dot{a}} + L_{ab\dot{a}}$ is just L_{3+3} , as can be seen from table 1.

We now follow the method in [14] to prove the self-dual property of L_{1+2+3} and follow the method in [10] to prove that the new non-covariant action gives field equations with 6d Lorentz invariance.

3.2.1 Self-duality in D=1+2+3

First, we rewrite the Lagrangian as

$$\begin{aligned} L_{1+2+3} &= \tilde{F}_{\dot{a}\dot{b}\dot{c}}(F^{\dot{a}\dot{b}\dot{c}} - \tilde{F}^{\dot{a}\dot{b}\dot{c}}) + 6\tilde{F}_{1a\dot{a}}(F^{1a\dot{a}} - \tilde{F}^{1a\dot{a}}) + 3\tilde{F}_{1\dot{a}\dot{b}}(F^{1\dot{a}\dot{b}} - \tilde{F}^{1\dot{a}\dot{b}}) \\ &= -3\tilde{F}_{1ab}F^{1ab} + 3F_{1ab}F^{1ab} - 3\tilde{F}_{a\dot{a}\dot{b}}F^{a\dot{a}\dot{b}} + 3F_{a\dot{a}\dot{b}}F^{a\dot{a}\dot{b}} - 3\tilde{F}_{ab\dot{a}}F^{ab\dot{a}} + 3F_{ab\dot{a}}F^{ab\dot{a}} \end{aligned} \quad (3.37)$$

The variation of the action S_{1+2+3} gives

$$\frac{\delta S_{1+2+3}}{\delta A_{1\dot{a}}} = -6(\partial_a \tilde{F}^{a1\dot{a}} + \partial_{\dot{b}} \tilde{F}^{\dot{b}1\dot{a}}) = 0 \quad (3.38)$$

which is identically zero. This means that terms involved $A_{1\dot{a}}$ only through total derivative terms and we have a **gauge symmetry**

$$\delta A_{1\dot{a}} = \Phi_{1\dot{a}} \quad (3.39)$$

for arbitrary functions $\Phi_{1\dot{a}}$.

Next, the field equation

$$0 = \frac{\delta S_{1+2+3}}{\delta A_{\dot{a}b}} = -3(\partial_{\dot{c}} \tilde{F}^{\dot{c}ab} + \partial_1 \tilde{F}^{1\dot{a}b} + \partial_a \tilde{F}^{a\dot{a}b}) - 6\partial_a \mathcal{F}^{a\dot{a}b} = -6\partial_a \mathcal{F}^{a\dot{a}b} \quad (3.40)$$

has solution

$$\mathcal{F}^{a\dot{a}b} = \epsilon^{1ab\dot{a}c} \partial_b \Phi_{1\dot{c}} \quad (3.41)$$

for arbitrary functions $\Phi_{1\dot{c}}$. Using the above **gauge symmetry** to completely remove $\Phi_{1\dot{c}}$ in $\mathcal{F}_{a\dot{a}b}$ we obtain a self-dual relation

$$\mathcal{F}_{a\dot{a}b} = 0 \quad (3.42)$$

To proceed we need to find more gauge symmetry. First, as term $\Phi_{1\dot{c}}$ is shown as $\partial_b \Phi_{1\dot{c}} \equiv \frac{\partial \Phi_{1\dot{c}}}{\partial x^b}$ in $\mathcal{F}^{a\dot{a}b}$ we have a furthermore symmetry

$$\delta A_{1\dot{a}} = W_{1\dot{a}}(x_1, x_{\dot{a}}) \quad (3.43)$$

in which $W_{1\dot{a}}(x_1, x_{\dot{a}})$ is an arbitrary function independent on the coordinate x_a . In short, after using the gauge symmetry to find a self-dual relation we still have above “residual gauge symmetries”. Next, as field $A_{\dot{a}b}$ only appears as $\partial_a A_{\dot{a}b}$ in $\mathcal{F}^{a\dot{a}b}$, therefore the previous results does not be modified under the variation

$$\delta A_{\dot{a}b} = W_{\dot{a}b}(x_1, x_{\dot{a}}) \quad (3.44)$$

in which $W_{\dot{a}b}(x_1, x_{\dot{a}})$ is an arbitrary function independent on the coordinate x_a . We will use the two **residual gauge symmetries** to find other self-duality relations. Note that these residual gauge symmetries do not spoil any of the self-duality conditions already satisfied.

Now, the field equation

$$\begin{aligned} 0 = \frac{\delta S_{1+2+3}}{\delta A_{\dot{a}a}} &= -6(\partial_1 \tilde{F}^{1a\dot{a}} + \partial_{\dot{b}} \tilde{F}^{\dot{b}a\dot{a}} + \partial_b \tilde{F}^{ba\dot{a}}) + 12(\partial_{\dot{b}} \mathcal{F}^{\dot{b}a\dot{a}} + \partial_b \mathcal{F}^{ba\dot{a}}) \\ &= 12(\partial_{\dot{b}} \mathcal{F}^{\dot{b}a\dot{a}} + \partial_b \mathcal{F}^{ba\dot{a}}) = \partial_b \mathcal{F}^{ba\dot{a}} \end{aligned} \quad (3.45)$$

tells us that $\mathcal{F}^{ba\dot{a}}$ is independent of the coordinate “ x_a ” .

In the same way, the field equation

$$0 = \frac{\delta S_{1+2+3}}{\delta A_{1a}} = -6(\partial_{\dot{a}} \tilde{F}^{\dot{a}1a} + \partial_b \tilde{F}^{b1a}) - 12\partial_b \mathcal{F}^{b1a} = -12\partial_b \mathcal{F}^{b1a} \quad (3.46)$$

tells us that \mathcal{F}^{1ab} is independent of the coordinate “ x_a ” .

Finally, using above properties the field equation

$$\begin{aligned} 0 = \frac{\delta S_{1+2+3}}{\delta A_{ab}} &= -3(\partial_1 \tilde{F}^{1ab} + \partial_{\dot{a}} \tilde{F}^{\dot{a}ab}) + 6(\partial_1 \mathcal{F}^{1ab} + \partial_{\dot{a}} \mathcal{F}^{\dot{a}ab}) \\ &= 6(\partial_1 \mathcal{F}^{1ab} + \partial_{\dot{a}} \mathcal{F}^{\dot{a}ab}) \end{aligned} \quad (3.47)$$

has solution

$$\mathcal{F}^{1ab} = \epsilon^{\dot{a}\dot{b}\dot{c}} \partial_{\dot{a}} W_{\dot{b}\dot{c}}(x_1, x_{\dot{a}}) \quad (3.48)$$

$$\mathcal{F}^{ab\dot{a}} = \epsilon^{\dot{a}\dot{b}\dot{c}} \left[-\partial_1 W_{\dot{b}\dot{c}}(x_1, x_{\dot{a}}) + \partial_{\dot{b}} W_{1\dot{c}}(x_1, x_{\dot{a}}) \right] \quad (3.49)$$

As $W_{\dot{b}\dot{c}}$ and $W_{1\dot{c}}$ are arbitrary functions independent on the coordinates “ x_a ” we can use the residual gauge symmetries to completely remove $W_{\dot{b}\dot{c}}$ and $W_{1\dot{c}}$. Thus we obtain the self-dual relations

$$\mathcal{F}_{1ab} = 0 \quad (3.50)$$

$$\mathcal{F}_{ab\dot{a}} = 0 \quad (3.51)$$

These complete the proof.

3.2.2 Lorentz Invariance in D=1+2+3

As covariant symmetry on 2D coordinates x_a and 3D coordinates $x_{\dot{a}}$ are manifest we only need to examine transformations (I) mixing x_1 with x_a , (II) mixing x_1 with $x_{\dot{a}}$ and (III) mixing x_a with $x_{\dot{a}}$.

(I) For the mixing x_1 with $x_{\dot{a}}$ we shall consider the transformation

$$\delta x^{\dot{a}} = \omega^{\dot{a}1} x_1 \equiv \Lambda^{\dot{a}} x_1, \quad (3.52)$$

$$\delta x^1 = \omega^{1\dot{a}} x_{\dot{a}} = -\Lambda^{\dot{a}} x_{\dot{a}} = -\Lambda \cdot x \quad (3.53)$$

Define

$$\Lambda \cdot L \equiv (\Lambda \cdot x) \partial_1 - x_1 (\Lambda \cdot \partial) \quad (3.54)$$

then we see that

$$\delta(F_{1ab} - \tilde{F}_{1ab}) = (\Lambda \cdot L)(F_{1ab} - \tilde{F}_{1ab}) - \Lambda^{\dot{a}}(F_{ab\dot{a}} - \tilde{F}_{ab\dot{a}}) \quad (3.55)$$

$$\delta(F_{1a\dot{b}} - \tilde{F}_{1a\dot{b}}) = (\Lambda \cdot L)(F_{1a\dot{b}} - \tilde{F}_{1a\dot{b}}) + \Lambda^{\dot{c}}(F_{\dot{c}ab} - \tilde{F}_{\dot{c}ab}) \quad (3.56)$$

$$\delta(F_{1\dot{a}b} - \tilde{F}_{1\dot{a}b}) = (\Lambda \cdot L)(F_{1\dot{a}b} - \tilde{F}_{1\dot{a}b}) + \Lambda^{\dot{c}}(F_{\dot{c}ab} - \tilde{F}_{\dot{c}ab}) \quad (3.57)$$

which are zero for self-dual theory and the non-covariant action gives field equation with 6d Lorentz transformation mixing x_1 with $x_{\dot{a}}$.

(II) For the mixing x_1 with x_a we shall consider the transformation

$$\delta x^a = \omega^{a1} x_1 \equiv \Lambda^a x_1, \quad (3.58)$$

$$\delta x^1 = \omega^{1a} x_a = -\Lambda^a x_a = -\Lambda \cdot x \quad (3.59)$$

Define

$$\Lambda \cdot L \equiv (\Lambda \cdot x) \partial_1 - x_1 (\Lambda \cdot \partial) \quad (3.60)$$

then we see that

$$\delta(F_{1ab} - \tilde{F}_{1ab}) = (\Lambda \cdot L)(F_{1ab} - \tilde{F}_{1ab}) \quad (3.61)$$

$$\delta(F_{1a\dot{a}} - \tilde{F}_{1a\dot{a}}) = (\Lambda \cdot L)(F_{1a\dot{a}} - \tilde{F}_{1a\dot{a}}) + \Lambda^c(F_{c\dot{a}b} - \tilde{F}_{c\dot{a}b}) \quad (3.62)$$

$$\delta(F_{1\dot{a}b} - \tilde{F}_{1\dot{a}b}) = (\Lambda \cdot L)(F_{1\dot{a}b} - \tilde{F}_{1\dot{a}b}) + \Lambda^c(F_{c\dot{a}b} - \tilde{F}_{c\dot{a}b}) \quad (3.63)$$

which are zero for self-dual theory and the non-covariant action gives field equation with 6d Lorentz transformation mixing x_1 with x_a .

(III) Finally, we consider the mixing x_a with $x_{\dot{a}}$. In this case the transformation is

$$\delta x_a = \Lambda_a^{\dot{a}} x_{\dot{a}} \quad (3.64)$$

$$\delta x_{\dot{a}} = \Lambda_{\dot{a}}^a x_a \quad (3.65)$$

Define

$$\Lambda \cdot L \equiv \Lambda^{a\dot{a}}(x_a \partial_{\dot{a}} - x_{\dot{a}} \partial_a) \quad (3.66)$$

then we see that

$$\delta(F_{1ab} - \tilde{F}_{1ab}) = (\Lambda \cdot L)(F_{1ab} - \tilde{F}_{1ab}) - \Lambda_a^{\dot{a}}(F_{1\dot{a}b} - F_{1\dot{a}b}) - \Lambda_b^{\dot{b}}(F_{1a\dot{b}} - F_{1a\dot{b}}) \quad (3.67)$$

$$\delta(F_{1a\dot{b}} - F_{1a\dot{b}}) = (\Lambda \cdot L)(F_{1a\dot{b}} - F_{1a\dot{b}}) - \Lambda_a^{\dot{a}}(F_{1\dot{a}b} - F_{1\dot{a}b}) - \Lambda_{\dot{b}}^b(F_{1ab} - F_{1ab}) \quad (3.68)$$

$$\delta(F_{1\dot{a}b} - F_{1\dot{a}b}) = (\Lambda \cdot L)(F_{1\dot{a}b} - F_{1\dot{a}b}) + \Lambda_{\dot{a}}^a(F_{1ab} - F_{1ab}) + \Lambda_{\dot{b}}^b(F_{1\dot{a}b} - F_{1\dot{a}b}) \quad (3.69)$$

which are zero for self-dual theory and the non-covariant action gives field equation with 6d Lorentz transformation mixing x_a with $x_{\dot{a}}$.

In summary, we have found the non-covariant action of self-dual 2-form in decomposition $D = 1 + 2 + 3$ and have checked that the non-covariant action gives field equation with 6d Lorentz transformation.

3.3 D=2+2+2

In the (2+2+2) decomposition the spacetime index A is decomposed as $A = (a, \dot{a}, \ddot{a})$, with $a = (1, 2)$, $\dot{a} = (3, 4)$ and $\ddot{a} = (5, 6)$. Now, from table 2 we see that $L_{ABC} = (L_{a\dot{a}b}, L_{a\ddot{a}b}, L_{ab\dot{a}}, L_{ab\ddot{a}}, L_{a\dot{a}\dot{b}}, L_{a\ddot{a}\dot{b}}, L_{a\dot{a}\ddot{a}}, L_{a\ddot{a}\ddot{a}})$. Then, in terms of L_{ABC} the Lagrangian can be expressed as

$$L_{2+2+2} = \sum L_{ab\dot{a}} + \sum L_{ab\ddot{a}} + \sum L_{a\dot{a}b} + \sum L_{a\ddot{a}b} \quad (3.70)$$

We now follow the method in [14] to prove the self-dual property of L_{2+2+2} and follow the method in [10] to prove that the new non-covariant action gives field equations with 6d Lorentz invariance.

3.3.1 Self-duality in D=2+2+2

First, we rewrite the Lagrangian as

$$\begin{aligned} L_{2+2+2} &= \tilde{F}_{ab\dot{a}}(F^{ab\dot{a}} - \tilde{F}^{ab\dot{a}}) + \tilde{F}_{ab\ddot{a}}(F^{ab\ddot{a}} - \tilde{F}^{ab\ddot{a}}) + \tilde{F}_{a\dot{a}b}(F^{a\dot{a}b} - \tilde{F}^{a\dot{a}b}) \\ &\quad + \tilde{F}_{a\ddot{a}b}(F^{a\ddot{a}b} - \tilde{F}^{a\ddot{a}b}) \\ &= -\tilde{F}_{\dot{a}\ddot{a}b}F^{\dot{a}\ddot{a}b} + F_{\dot{a}\ddot{a}b}F^{\dot{a}\ddot{a}b} - \tilde{F}_{\dot{a}b\ddot{a}}F^{\dot{a}b\ddot{a}} + F_{\dot{a}b\ddot{a}}F^{\dot{a}b\ddot{a}} - \tilde{F}_{a\dot{a}\ddot{b}}F^{a\dot{a}\ddot{b}} + F_{a\dot{a}\ddot{b}}F^{a\dot{a}\ddot{b}} \\ &\quad - \tilde{F}_{a\ddot{a}\dot{b}}F^{a\ddot{a}\dot{b}} + F_{a\ddot{a}\dot{b}}F^{a\ddot{a}\dot{b}} \end{aligned} \quad (3.71)$$

The variation of the action S_{2+2+3} gives

$$\frac{\delta S_{2+2+2}}{\delta A_{ab}} = -(\partial_{\dot{a}}\tilde{F}^{\dot{a}ab} + \partial_{\ddot{a}}\tilde{F}^{\ddot{a}ab}) = 0 \quad (3.72)$$

which is identically zero and terms involved A_{ab} only through total derivative terms. Thus, as before, we have a **gauge symmetry**

$$\delta A_{ab} = \Phi_{ab} \quad (3.73)$$

for arbitrary functions Φ_{ab} .

Next, the field equation

$$0 = \frac{\delta S_{2+2+2}}{\delta A_{\dot{a}\dot{b}}} = -(\partial_a \tilde{F}^{a\dot{a}\dot{b}} + \partial_{\dot{a}} \tilde{F}^{\dot{a}a\dot{b}}) - 2\partial_{\dot{a}} \mathcal{F}^{\dot{a}\dot{a}\dot{b}} = -2\partial_{\dot{a}} \mathcal{F}^{\dot{a}\dot{a}\dot{b}} \quad (3.74)$$

has solution

$$\mathcal{F}^{\dot{a}\dot{b}\dot{c}} = \epsilon^{ab\dot{a}\dot{b}\dot{c}} \partial_{\dot{b}} \Phi_{ab} \quad (3.75)$$

for arbitrary functions Φ^{ab} . As before, we can now use the gauge symmetry of A_{ab} to reduce $\mathcal{F}_{\dot{a}\dot{b}\dot{c}}$ to be zero

$$\mathcal{F}_{\dot{a}\dot{b}\dot{c}} = 0 \quad (3.76)$$

and find the first self-dual relation.

To proceed we need to find more gauge symmetry. First, as term Φ_{ab} is shown as $\partial_{\dot{b}} \Phi_{ab}$ in $\mathcal{F}^{\dot{a}\dot{b}\dot{c}}$ we have a furthermore symmetry

$$\delta A_{ab} = W_{ab}(x_a, x_{\dot{a}}) \quad (3.77)$$

in which $W_{ab}(x_a, x_{\dot{a}})$ is an arbitrary function independing on the coordinate $x_{\dot{a}}$. In short, after using the gauge symmetry to find a self-dual relation we still have above residual gauge symmetries. Next, as field $A_{\dot{a}\dot{b}}$ appears only as $\partial_{\dot{a}} A_{\dot{a}\dot{b}}$ in $\mathcal{F}^{\dot{a}\dot{b}\dot{c}}$ the above relations do not be modified under the variation

$$\delta A_{\dot{a}\dot{b}} = W_{\dot{a}\dot{b}}(x_a, x_{\dot{a}}) \quad (3.78)$$

in which $W_{\dot{a}\dot{b}}(x_a, x_{\dot{a}})$ is an arbitrary function independing on the coordinate x_a . We need the above two **residual gauge symmetries** to find other self-duality relations in below. Note that these residual gauge symmetries do not spoil any of the self-duality conditions already satisfied.

Now, consider the field equation

$$0 = \frac{\delta S_{2+2+2}}{\delta A_{a\dot{a}}} = -2(\partial_b \tilde{F}^{ba\dot{a}} + \partial_{\dot{b}} \tilde{F}^{\dot{b}a\dot{a}} + \partial_{\dot{a}} \tilde{F}^{\dot{a}a\dot{a}}) + 2\partial_{\dot{a}} \mathcal{F}^{\dot{a}\dot{a}\dot{a}} = 2\partial_{\dot{a}} \mathcal{F}^{\dot{a}\dot{a}\dot{a}} \quad (3.79)$$

which has solution

$$\mathcal{F}_{a\dot{a}\dot{b}} = \epsilon_{ab\dot{a}\dot{b}\dot{c}} \partial^{\dot{c}} \Phi^{bb} \quad (3.80)$$

We can now follow [14] to find another self-dual relation. First, taking the Hodge-dual of both sides in above equation we find that

$$\mathcal{F}_{a\dot{a}\dot{b}} = \partial_{\dot{a}} \Phi_{a\dot{b}} \quad (3.81)$$

Identifying above two solutions leads to

$$\partial_{\bar{a}}\Phi_{a\bar{a}} = \epsilon_{ab\bar{a}\bar{b}}\partial^{\bar{b}}\Phi^{bb} \quad (3.82)$$

Acting $\partial^{\bar{a}}$ a on both sides gives

$$\partial^{\bar{a}}\partial_{\bar{a}}\Phi_{a\bar{a}} = 0 \quad (3.83)$$

Following [14], imposing the boundary condition that the regular field $\Phi_{a\bar{a}}$ be vanished at infinities will lead to the unique solution $\Phi_{a\bar{a}} = 0$ and we arrive at the self-duality conditions

$$\mathcal{F}_{a\bar{a}\bar{a}} = 0 \quad (3.84)$$

To proceed we need to find one more gauge symmetry. As term $\Phi_{a\bar{a}}$ is shown as $\partial_{\bar{a}}\Phi_{a\bar{a}}$ in $\mathcal{F}^{a\bar{a}\bar{a}}$ we have a furthermore symmetry

$$\delta A_{a\bar{a}} = W_{a\bar{a}}(x_a, x_{\bar{a}}) \quad (3.85)$$

in which $W_{a\bar{a}}(x_a, x_{\bar{a}})$ is an arbitrary function independing on the coordinate $x_{\bar{a}}$. In short, after using the gauge symmetry to find a self-dual relation we still have above residual gauge symmetries. We need the above **residual gauge symmetries** to find other self-duality relations in below. Note that these residual gauge symmetries do not spoil any of the self-duality conditions already satisfied.

Use the found self-dual relation the field equation becomes

$$\begin{aligned} 0 = \frac{\delta S_{2+2+2}}{\delta A_{a\bar{a}}} &= -2(\partial_{\bar{b}}\tilde{F}^{\bar{b}a\bar{a}} + \partial_{\bar{b}}\tilde{F}^{\bar{b}a\bar{a}} + \partial_a\tilde{F}^{a\bar{a}\bar{a}}) + 4\partial_{\bar{b}}\mathcal{F}^{\bar{b}b\bar{a}} + 4\partial_{\bar{b}}\mathcal{F}^{\bar{b}b\bar{a}} + 4\partial_a\mathcal{F}^{a\bar{b}\bar{a}} \\ &= 4\partial_{\bar{b}}\mathcal{F}^{\bar{b}b\bar{a}} \end{aligned} \quad (3.86)$$

Thus $\mathcal{F}^{\bar{a}\bar{a}\bar{b}}$ is independent of coordinate $x_{\bar{a}}$

In a same way, the field equation becomes

$$\begin{aligned} 0 = \frac{\delta S_{2+2+2}}{\delta A_{a\bar{a}}} &= -2(\partial_{\bar{b}}\tilde{F}^{\bar{b}a\bar{a}} + \partial_{\bar{b}}\tilde{F}^{\bar{b}a\bar{a}} + \partial_b\tilde{F}^{ba\bar{a}}) + 4\partial_{\bar{b}}\mathcal{F}^{\bar{b}a\bar{a}} + 2\partial_{\bar{a}}\mathcal{F}^{\bar{a}a\bar{a}} \\ &= 4\partial_{\bar{b}}\mathcal{F}^{\bar{b}a\bar{a}} \end{aligned} \quad (3.87)$$

Thus $\mathcal{F}^{a\bar{a}\bar{b}}$ is independent of coordinate $x_{\bar{a}}$

Use above property of independent of coordinate $x_{\bar{a}}$ the final field equation

$$\begin{aligned} 0 = \frac{\delta S_{2+2+2}}{\delta A_{\bar{a}\bar{b}}} &= -(\partial_{\bar{a}}\tilde{F}^{\bar{a}a\bar{b}} + \partial_a\tilde{F}^{a\bar{a}\bar{b}}) + 2\partial_{\bar{a}}\mathcal{F}^{\bar{a}\bar{a}\bar{b}} + 2\partial_a\mathcal{F}^{a\bar{a}\bar{b}} \\ &= 2\partial_{\bar{a}}\mathcal{F}^{\bar{a}\bar{a}\bar{b}} + 2\partial_a\mathcal{F}^{a\bar{a}\bar{b}} \end{aligned} \quad (3.88)$$

gives the solution

$$\mathcal{F}^{a\ddot{a}\ddot{b}} = \epsilon^{ab\dot{a}\dot{b}\ddot{a}\ddot{b}}[\partial_{\dot{b}}W_{\dot{a}\dot{b}} + \partial_{\dot{a}}W_{\dot{b}\dot{b}}] \quad (3.89)$$

$$\mathcal{F}^{\dot{a}\ddot{a}\ddot{b}} = \epsilon^{ab\dot{a}\dot{b}\ddot{a}\ddot{b}}[\partial_aW_{\dot{b}\dot{b}} + \partial_{\dot{b}}W_{ab}] \quad (3.90)$$

As $W_{\dot{a}\dot{b}}$, $W_{a\dot{a}}$ and W_{ab} are arbitrary functions independent on the coordinates “ $x_{\dot{a}}$ ” we can use the “residual gauge symmetries” to completely remove them. Thus we obtain the self-dual relations

$$\mathcal{F}^{a\ddot{a}\ddot{b}} = 0 \quad (3.91)$$

$$\mathcal{F}^{\dot{a}\ddot{a}\ddot{b}} = 0 \quad (3.92)$$

These complete the proof.

3.3.2 Lorentz Invariance in D=2+2+2

As covariant symmetry on 2D coordinates x_a , 2D coordinates $x_{\dot{a}}$ and 2D coordinates $x_{\ddot{a}}$ are manifest we only need to examine transformations (I) mixing x_a with $x_{\dot{a}}$, (II) mixing $x_{\dot{a}}$ with $x_{\ddot{a}}$ and (III) mixing x_a with $x_{\ddot{a}}$.

We consider the mixing x_a with $x_{\dot{a}}$. In this case the transformation is

$$\delta x_a = \Lambda_a^{\dot{a}} x_{\dot{a}} \quad (3.93)$$

$$\delta x_{\dot{a}} = \Lambda_{\dot{a}}^a x_a \quad (3.94)$$

Define

$$\Lambda \cdot L \equiv \Lambda^{a\dot{a}}(x_a \partial_{\dot{a}} - x_{\dot{a}} \partial_a) \quad (3.95)$$

then

$$\delta F_{ab\dot{a}} = (\Lambda \cdot L)F_{ab\dot{a}} - \Lambda_a^{\dot{b}} F_{\dot{b}\dot{b}\dot{a}} - \Lambda_b^{\dot{b}} F_{\dot{a}\dot{b}\dot{a}} \quad (3.96)$$

$$\delta F_{ab\ddot{a}} = (\Lambda \cdot L)F_{ab\ddot{a}} - \Lambda_a^{\dot{b}} F_{\dot{b}\dot{b}\ddot{a}} - \Lambda_b^{\dot{b}} F_{\dot{a}\dot{b}\ddot{a}} \quad (3.97)$$

$$\delta F_{a\dot{a}\ddot{b}} = (\Lambda \cdot L)F_{a\dot{a}\ddot{b}} + \Lambda_a^{\dot{b}} F_{\dot{a}\dot{b}\ddot{b}} + \Lambda_{\dot{a}}^{\dot{b}} F_{a\dot{b}\ddot{b}} \quad (3.98)$$

$$\delta F_{a\dot{a}\ddot{a}} = (\Lambda \cdot L)F_{a\dot{a}\ddot{a}} - \Lambda_a^{\dot{b}} F_{\dot{b}\dot{b}\ddot{a}} - \Lambda_{\dot{a}}^{\dot{b}} F_{a\dot{b}\ddot{a}} \quad (3.99)$$

$$\delta F_{a\ddot{a}\ddot{b}} = (\Lambda \cdot L)F_{a\ddot{a}\ddot{b}} - \Lambda_a^{\dot{a}} F_{\dot{a}\ddot{a}\ddot{b}} \quad (3.100)$$

$$\delta F_{\dot{a}\ddot{a}\ddot{b}} = (\Lambda \cdot L)F_{\dot{a}\ddot{a}\ddot{b}} + \Lambda_{\dot{a}}^a F_{a\ddot{a}\ddot{b}} \quad (3.101)$$

$$\delta F_{\dot{a}\dot{b}\ddot{a}} = (\Lambda \cdot L)F_{\dot{a}\dot{b}\ddot{a}} + \Lambda_{\dot{a}}^a F_{a\dot{b}\ddot{a}} + \Lambda_{\dot{b}}^a F_{\dot{a}a\ddot{a}} \quad (3.102)$$

Use above transformation we can calculate the transformations of $\tilde{F}_{ab\dot{a}}$, $\tilde{F}_{ab\ddot{a}}$, $\tilde{F}_{a\dot{a}\dot{b}}$ and $\tilde{F}_{a\dot{a}\ddot{a}}$. Then we see that

$$\delta(F_{ab\dot{a}} - \tilde{F}_{ab\dot{a}}) = (\Lambda \cdot L)(F_{ab\dot{a}} - \tilde{F}_{ab\dot{a}}) - \Lambda_a^{\dot{b}}(F_{bb\dot{a}} - \tilde{F}_{bb\dot{a}}) - \Lambda_b^{\dot{a}}(F_{ab\dot{a}} - \tilde{F}_{ab\dot{a}}) \quad (3.103)$$

$$\delta(F_{ab\ddot{a}} - \tilde{F}_{ab\ddot{a}}) = (\Lambda \cdot L)(F_{ab\ddot{a}} - \tilde{F}_{ab\ddot{a}}) - \Lambda_a^{\dot{b}}(F_{bb\ddot{a}} - \tilde{F}_{bb\ddot{a}}) - \Lambda_b^{\ddot{a}}(F_{ab\ddot{a}} - \tilde{F}_{ab\ddot{a}}) \quad (3.104)$$

$$\delta(F_{a\dot{a}\dot{b}} - \tilde{F}_{a\dot{a}\dot{b}}) = (\Lambda \cdot L)(F_{a\dot{a}\dot{b}} - \tilde{F}_{a\dot{a}\dot{b}}) + \Lambda_a^{\dot{b}}(F_{ab\dot{b}} - \tilde{F}_{ab\dot{b}}) + \Lambda_b^{\dot{a}}(F_{a\dot{a}\dot{b}} - \tilde{F}_{a\dot{a}\dot{b}}) \quad (3.105)$$

$$\delta(F_{a\dot{a}\ddot{a}} - \tilde{F}_{a\dot{a}\ddot{a}}) = (\Lambda \cdot L)(F_{a\dot{a}\ddot{a}} - \tilde{F}_{a\dot{a}\ddot{a}}) - \Lambda_a^{\dot{b}}(F_{b\dot{a}\ddot{a}} - \tilde{F}_{b\dot{a}\ddot{a}}) - \Lambda_b^{\ddot{a}}(F_{a\dot{a}\ddot{a}} - \tilde{F}_{a\dot{a}\ddot{a}}) \quad (3.106)$$

which are zero for self-dual theory and the non-covariant action gives field equations with 6d Lorentz transformation mixing x_a with $x_{\dot{a}}$. The transformation mixing x_a with $x_{\ddot{a}}$ and mixing $x_{\dot{a}}$ with $x_{\ddot{a}}$ have the similar results.

In summary, we have found the non-covariant action of self-dual 2-form in decomposition $D = 2 + 2 + 2$ and checked that the non-covariant action gives field equations with 6d Lorentz transformation.

4 Other Decomposition and Spacetime

4.1 Other Decomposition : D=1+1+1+3

Besides the decomposition in section 2 and section 3 there are many other possible decompositions. We will in this subsection see that it is possible to found the Lagrangian of self-dual 2 form in decomposition : $D = 1 + 1 + 1 + 3$.

In the (1+1+1+3) decomposition the spacetime index A is decomposed as $A = (1, 2, 3, \dot{a})$, with $\dot{a} = (4, 5, 6)$, and $L_{ABC} = (L_{123}, L_{12\dot{a}}, L_{13\dot{a}}, L_{1\dot{a}\dot{b}}, L_{\dot{a}\dot{b}\dot{c}}, L_{2\dot{a}\dot{b}}, L_{3\dot{a}\dot{b}}, L_{23\dot{a}})$. From table 3 it is easy to see that, in terms of L_{ABC} , the Lagrangian can be expressed as

$$L_{1+1+1+3} = 6L_{123} + 6 \sum L_{12\dot{a}} + 6 \sum L_{13\dot{a}} + \frac{3}{2} \sum L_{1\dot{a}\dot{b}} + \frac{6}{2} \sum L_{23\dot{a}} \quad (4.1)$$

Other choices will becomes the decompositions in section 2 or section 3, as can be seen from table 1 and table 2. Note that the factor 6 or 3 in (4.1) is to count the number of possible permutation and the factor $\frac{1}{2}$ in (4.1) reveals that fact we have counted both the left-hand side and right-hand side in table 3.

Table 3: Lagrangian in decompositions: $D = 1 + 1 + 1 + 3$.

$$D=1+1+1+3$$

| | | | |
|-----|-----------------------|-----|---------------------------------------|
| 123 | L_{123} | 456 | $\mathcal{L}_{\dot{a}\dot{b}\dot{c}}$ |
| 124 | $L_{12\dot{a}}$ | 356 | $\mathcal{L}_{3\dot{a}\dot{b}}$ |
| 125 | | 346 | |
| 126 | | 345 | |
| 134 | $L_{13\dot{a}}$ | 256 | $\mathcal{L}_{2\dot{a}\dot{b}}$ |
| 135 | | 246 | |
| 136 | | 245 | |
| 145 | $L_{1\dot{a}\dot{b}}$ | 236 | $L_{23\dot{a}}$ |
| 146 | | 235 | |
| 156 | | 234 | |

We follow the method in [14] to prove the self-dual property of $L_{1+1+1+3}$ and follow the method in [10] to prove that the new non-covariant action gives field equations with 6d Lorentz invariance.

4.1.1 Self-duality in $D=1+1+1+3$

First, we rewrite the Lagrangian as

$$\begin{aligned}
L_{1+1+1+3} &= 6\tilde{F}_{123}(F^{123} - \tilde{F}^{123}) + 6\tilde{F}_{12\dot{a}}(F^{12\dot{a}} - \tilde{F}^{12\dot{a}}) + 6\tilde{F}_{13\dot{a}}(F^{13\dot{a}} - \tilde{F}^{13\dot{a}}) \\
&\quad + \frac{3}{2}\tilde{F}_{1\dot{a}\dot{b}}(F^{1\dot{a}\dot{b}} - \tilde{F}^{1\dot{a}\dot{b}}) + \frac{6}{2}\tilde{F}_{23\dot{a}}(F^{23\dot{a}} - \tilde{F}^{23\dot{a}}) \\
&= -\tilde{F}_{\dot{a}\dot{b}\dot{c}}F^{\dot{a}\dot{b}\dot{c}} + F_{\dot{a}\dot{b}\dot{c}}\tilde{F}^{\dot{a}\dot{b}\dot{c}} - 3\tilde{F}_{3\dot{a}\dot{b}}F^{3\dot{a}\dot{b}} + 3F_{3\dot{a}\dot{b}}\tilde{F}^{3\dot{a}\dot{b}} - 3\tilde{F}_{2\dot{a}\dot{b}}F^{2\dot{a}\dot{b}} + 3F_{2\dot{a}\dot{b}}\tilde{F}^{2\dot{a}\dot{b}} \\
&\quad - 3\tilde{F}_{23\dot{a}}F^{23\dot{a}} + 3F_{23\dot{a}}\tilde{F}^{23\dot{a}} - \frac{3}{2}\tilde{F}_{1\dot{a}\dot{b}}F^{1\dot{a}\dot{b}} + \frac{3}{2}F_{1\dot{a}\dot{b}}\tilde{F}^{1\dot{a}\dot{b}}
\end{aligned} \tag{4.2}$$

The variation of the action $S_{1+1+1+3}$ gives

$$\frac{\delta S_{1+1+1+3}}{\delta A_{12}} = -6\partial_3\tilde{F}^{312} - 6\partial_{\dot{a}}\tilde{F}^{\dot{a}12} = 0, \tag{4.3}$$

$$\frac{\delta S_{1+1+1+3}}{\delta A_{13}} = -6\partial_2\tilde{F}^{213} - 6\partial_{\dot{a}}\tilde{F}^{\dot{a}13} = 0 \tag{4.4}$$

which are identically zero. This means that terms involved A_{12} and A_{13} only through total derivative terms and we have a gauge symmetry

$$\delta A_{12} = \Phi_{12} \tag{4.5}$$

$$\delta A_{13} = \Phi_{13} \tag{4.6}$$

for arbitrary functions Φ_{12} and Φ_{13} .

We also have following two field equations

$$0 = \frac{\delta S_{1+1+1+3}}{\delta A_{1\dot{a}}} = -6(\partial_2 \tilde{F}^{21\dot{a}} + \partial_3 \tilde{F}^{31\dot{a}} + \partial_b \tilde{F}^{b1\dot{a}}) + 6\partial_b \mathcal{F}^{b1\dot{a}} = 6\partial_b \mathcal{F}^{b1\dot{a}} \quad (4.7)$$

$$0 = \frac{\delta S_{1+1+1+3}}{\delta A_{23}} = -6(\partial_1 \tilde{F}^{123} + \partial_a \tilde{F}^{a23}) + 6\partial_a \mathcal{F}^{a23} = 6\partial_a \mathcal{F}^{a23} \quad (4.8)$$

which imply the solutions

$$\mathcal{F}^{1\dot{a}b} = \epsilon^{123\dot{a}bc} \partial_c \Phi_{23} \quad (4.9)$$

$$\mathcal{F}^{23\dot{a}} = \epsilon^{123\dot{a}bc} \partial_b \Phi_{1c} \quad (4.10)$$

Take the Hodge-dual of the first solution and compare it with the second solution we find that

$$\partial_a \Phi_{1b} = \epsilon_{123\dot{a}bc} \partial^{\dot{c}} \Phi_{23} \quad (4.11)$$

After acting $\partial^{\dot{a}}$ on both sides we find that

$$\partial^{\dot{a}} \partial_a \Phi_{1b} = 0 \quad (4.12)$$

As described in [14], imposing the boundary condition that the regular field Φ_{1b} be vanished at infinities leads to the unique solution $\Phi_{1\dot{a}} = 0$. Then we arrive at the self-duality conditions

$$\mathcal{F}_{23\dot{a}} = 0 \quad (4.13)$$

In a similar way we can also find another self-duality conditions

$$\mathcal{F}_{1\dot{a}b} = 0 \quad (4.14)$$

Using above result the field equation

$$\begin{aligned} 0 = \frac{\delta S_{1+1+1+3}}{\delta A_{2\dot{a}}} &= -6(\partial_1 \tilde{F}^{12\dot{a}} + \partial_3 \tilde{F}^{32\dot{a}} + \partial_b \tilde{F}^{b2\dot{a}}) + 6\partial_3 \mathcal{F}^{32\dot{a}} + 12\partial_b \mathcal{F}^{b2\dot{a}} \\ &= 6\partial_3 \mathcal{F}^{32\dot{a}} + 12\partial_b \mathcal{F}^{b2\dot{a}} = 12\partial_b \mathcal{F}^{b2\dot{a}} \end{aligned} \quad (4.15)$$

implies

$$\mathcal{F}_{2\dot{a}b} = \epsilon_{123\dot{a}bc} \partial^c \Phi_{13} \quad (4.16)$$

Thus, use the gauge symmetry $\delta A_{13} = \Phi_{13}$ we can completely remove the function Φ_{13} and find the self-dual relation.

$$\mathcal{F}_{2\dot{a}b} = 0 \quad (4.17)$$

In the same way, the field equation

$$\begin{aligned} 0 = \frac{\delta S_{1+1+1+3}}{\delta A_{3\dot{a}}} &= -6(\partial_1 \tilde{F}^{13\dot{a}} + \partial_2 \tilde{F}^{23\dot{a}} + \partial_2 \tilde{F}^{23\dot{a}}) + 6\partial_2 \mathcal{F}^{23\dot{a}} + 12\partial_{\dot{b}} \mathcal{F}^{b3\dot{a}} \\ &= 6\partial_2 \mathcal{F}^{23\dot{a}} + 12\partial_{\dot{b}} \mathcal{F}^{b3\dot{a}} = 12\partial_{\dot{b}} \mathcal{F}^{b3\dot{a}} \end{aligned} \quad (4.18)$$

implies

$$\mathcal{F}_{3\dot{a}\dot{b}} = \epsilon_{123\dot{a}\dot{b}\dot{c}} \partial^{\dot{c}} \Phi_{12} \quad (4.19)$$

Thus, use the gauge symmetry $\delta A_{12} = \Phi_{12}$ we can completely remove the function Φ_{12} and find the self-dual relation.

$$\mathcal{F}_{3\dot{a}\dot{b}} = 0 \quad (4.20)$$

Finally, through the calculation the field equation becomes

$$\begin{aligned} 0 = \frac{\delta S_{1+1+1+3}}{\delta A_{\dot{a}\dot{b}}} &= -3(\partial_{\dot{c}} \tilde{F}^{\dot{c}\dot{a}\dot{b}} + \partial_3 \tilde{F}^{3\dot{a}\dot{b}} + \partial_2 \tilde{F}^{2\dot{a}\dot{b}} + \partial_1 \tilde{F}^{1\dot{a}\dot{b}}) \\ &\quad + 6\partial_{\dot{c}} \mathcal{F}^{\dot{c}\dot{a}\dot{b}} + 6\partial_3 \mathcal{F}^{3\dot{a}\dot{b}} + 6\partial_2 \mathcal{F}^{2\dot{a}\dot{b}} + 3\partial_1 \tilde{F}^{1\dot{a}\dot{b}} \\ &= 6\partial_{\dot{c}} \mathcal{F}^{\dot{c}\dot{a}\dot{b}} + 6\partial_3 \mathcal{F}^{3\dot{a}\dot{b}} + 6\partial_2 \mathcal{F}^{2\dot{a}\dot{b}} + 3\partial_1 \tilde{F}^{1\dot{a}\dot{b}} = 6\partial_{\dot{c}} \mathcal{F}^{\dot{c}\dot{a}\dot{b}} \end{aligned} \quad (4.21)$$

in which we have used the found self-duality relations. Above field equation has solution

$$\mathcal{F}_{\dot{a}\dot{b}\dot{c}} = \epsilon_{\dot{a}\dot{b}\dot{c}} \Phi(x^1, x^2, x^3) \quad (4.22)$$

where $\Phi(x^1, x^2, x^3)$ is independent of the coordinates $x^{\dot{a}}$. As $\Phi(x^1, x^2, x^3)$ can be written as $\partial_i f^i(x^1, x^2, x^3)$, with $i = 1, 2, 3$, the function $\Phi(x^1, x^2, x^3)$ can be absorbed by a field redefinition $A_{ij} \rightarrow A_{ij} + \epsilon_{ijk} f^k(x^1, x^2, x^3)$. Thus, we have another self-duality condition

$$\mathcal{F}_{\dot{a}\dot{b}\dot{c}} = 0 \quad (4.23)$$

Together with other self-duality conditions we have found all self-duality conditions and complete the proof.

4.1.2 Lorentz Invariance in D=1+1+1+3

As covariant symmetry on 2D coordinates $x_{\dot{a}}$ is manifest we only need to examine transformations (I) mixing x_1 with x_2 and (II) mixing x_1 with $x_{\dot{a}}$. Other transformations needed to examine are just the exchange of $1 \leftrightarrow 2$, $1 \leftrightarrow 3$ or $2 \leftrightarrow 3$.

(I) For the mixing x_1 with $x_{\dot{a}}$ we shall consider the transformation

$$\delta x^{\dot{a}} = \omega^{\dot{a}1} x_1 \equiv \Lambda^{\dot{a}} x_1, \quad (4.24)$$

$$\delta x^1 = \omega^{1\dot{a}} x_{\dot{a}} = -\Lambda^{\dot{a}} x_{\dot{a}} = -\Lambda \cdot x \quad (4.25)$$

Define

$$\Lambda \cdot L \equiv (\Lambda \cdot x) \partial_1 - x_1 (\Lambda \cdot \partial) \quad (4.26)$$

then we see that

$$\delta(F_{123} - \tilde{F}_{123}) = (\Lambda \cdot L)(F_{123} - \tilde{F}_{123}) + \Lambda^{\dot{a}}(F_{\dot{a}23} - \tilde{F}_{\dot{a}23}) \quad (4.27)$$

$$\delta(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) = (\Lambda \cdot L)(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) + \Lambda^{\dot{b}}(F_{\dot{b}2\dot{a}} - \tilde{F}_{\dot{b}2\dot{a}}) \quad (4.28)$$

$$\delta(F_{13\dot{a}} - \tilde{F}_{13\dot{a}}) = (\Lambda \cdot L)(F_{13\dot{a}} - \tilde{F}_{13\dot{a}}) + \Lambda^{\dot{b}}(F_{\dot{b}3\dot{a}} - \tilde{F}_{\dot{b}3\dot{a}}) \quad (4.29)$$

$$\delta(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) = (\Lambda \cdot L)(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) + \Lambda^{\dot{c}}(F_{\dot{c}\dot{a}\dot{b}} - \tilde{F}_{\dot{c}\dot{a}\dot{b}}) \quad (4.30)$$

which are zero for self-dual theory and the non-covariant action gives field equations with 6d Lorentz transformation mixing x_1 with x_a .

(II) For the mixing x_1 with x_2 we shall consider the transformation

$$\delta x^1 = \omega^{12} x_2 \equiv \Lambda x_2, \quad (4.31)$$

$$\delta x^2 = \omega^{21} x_1 = -\Lambda x_1 \quad (4.32)$$

Define

$$\Lambda \cdot L \equiv \Lambda x_2 \partial_1 - x_2 \Lambda \partial_1 \quad (4.33)$$

then we see that

$$\delta(F_{123} - \tilde{F}_{123}) = (\Lambda \cdot L)(F_{123} - \tilde{F}_{123}) \quad (4.34)$$

$$\delta(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) = (\Lambda \cdot L)(F_{12\dot{a}} - \tilde{F}_{12\dot{a}}) \quad (4.35)$$

$$\delta(F_{13\dot{a}} - \tilde{F}_{13\dot{a}}) = (\Lambda \cdot L)(F_{13\dot{a}} - \tilde{F}_{13\dot{a}}) - \Lambda(F_{23\dot{a}} - \tilde{F}_{23\dot{a}}) \quad (4.36)$$

$$\delta(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) = (\Lambda \cdot L)(F_{1\dot{a}\dot{b}} - \tilde{F}_{1\dot{a}\dot{b}}) - \Lambda(F_{2\dot{a}\dot{b}} - \tilde{F}_{2\dot{a}\dot{b}}) \quad (4.37)$$

which are zero for self-dual theory and the non-covariant action gives field equations with 6d Lorentz transformation mixing x_1 with x_2 .

In summary, we have found the non-covariant action of self-dual 2-form in decomposition $D = 1 + 1 + 1 + 3$ and checked that the non-covariant action gives field equations with 6d Lorentz transformation.

4.2 Decomposition in Other Spacetime

We have seen that the Lagrangian of self-dual 2-form gauge field could be expressed as many different formulations. The extension to other form is straightforward.

For example, the self-dual 4-form gauge field in 10 D can be decomposed as $D = D_1 + D_2$, which is proved in [14]. For the decompositions $D = 1 + 9$, see table 4, the Lagrangian can be expressed as

$$L_{1+9} = -\frac{1}{4} \sum L_{1\dot{a}bcd} \quad (4.38)$$

with $\dot{a} = (2, \dots, 10)$.

Table 4: Lagrangian in decompositions: $D = 1 + 9$ and $D = 1 + 1 + 8$.

| D=1+9 | | D=1+1+8 | |
|-------------------|-------------------------|-------------------|-------------------------|
| $L_{1\dot{a}bcd}$ | $L_{\dot{a}bcd\dot{e}}$ | $L_{12\dot{a}bc}$ | $L_{\dot{a}bcd\dot{e}}$ |
| | | $L_{1\dot{a}bcd}$ | $L_{2\dot{a}bcd}$ |

The decomposition in $D = D_1 + D_2 + D_3$ can also be performed as before. In the case of $D = 1 + 1 + 8$ then, as that in section 3.1, the spacetime index A is decomposed as $A = (1, 2, \dot{a})$, with $\dot{a} = (3, \dots, 10)$, and $L_{ABCE F} = (L_{12\dot{a}bc}, L_{\dot{a}bcd\dot{e}}, L_{1\dot{a}bcd}, L_{2\dot{a}bcd})$. From table 4 we see that Lagrangian can be expressed as

$$L_{1+1+8} = -\frac{1}{4} \left(6 \sum L_{12\dot{a}bc} + 4 \sum L_{2\dot{a}bcd} \right) \quad (4.39)$$

Choosing $L_{12\dot{a}bc} + L_{2\dot{a}bcd}$ is just the decomposition $D = 1 + 9$ in [14]. The proof of self-dual relation is the same as those in section 3.1.

In conclusion, there are many different formulations of the self-dual gauge field. As the decomposition has many kind it seems not easy to provide a general proof of the self-dual relation in there and we have used some examples to illuminate the property.

5 Conclusion

In this paper we have first reviewed the Lagrangian of self-dual gauge theory in various non-covariant formulations. Then, we see a simple rule in there and use it to present some new Lagrangian of non-covariant forms of self-dual gauge theory. We see that the existence of gauge symmetry $\delta A = \Phi$ in the Lagrangian play important role of the self-dual relation. Using this as the guiding principle we have found many different formulations. It is interesting to see that in some cases it remains only one possible choice for the specified decomposition. Especially, we have followed the prescription in [14] to prove the self-dual property in the new Lagrangian in a detailed way. We

also have followed the method of Perry and Schwarz in to show that these new non-covariant actions give field equations with 6d Lorentz invariance.

Finally, the covariant form in each decomposition may be found by following the method in [9,10,13] and we leave the study in future research. It also remains to see whether the non-abelian self-dual gauge theory in 1+5 dimension [15] could be decomposed in other way.

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APPENDIX

A Coordinate Transformation

In this appendix we evaluate in detail the Lorentz transformation of 2-form field strength.

Under the coordinate transformation : $x_a \rightarrow x_a + \delta x_a$ we consider the tensor field $F_{\bar{M}\bar{N}\bar{P}}(x_a)$ defined by

$$\begin{aligned} F_{MNP}(x_a) &\rightarrow F_{\bar{M}\bar{N}\bar{P}}(x_a + \delta x_a) \equiv \frac{\partial x^Q}{\partial \bar{x}^M} \frac{\partial x^R}{\partial \bar{x}^N} \frac{\partial x^S}{\partial \bar{x}^P} F_{QRS}(x_a + \delta x_a) \\ &\approx F_{MNP}(x_a + \delta x_a) + \frac{\partial x^Q}{\partial \bar{x}^M} \frac{\partial x^R}{\partial \bar{x}^N} \frac{\partial x^S}{\partial \bar{x}^P} F_{QRS}(x_a) \end{aligned} \quad (\text{A.1})$$

For the transformation mixing between x_1 with x_μ ($\mu \neq 1$) the relation $\delta x_a = \omega_{ab}x^b$ leads to $\delta x_1 = -\Lambda_\mu x^\mu$ and $\delta x_\mu = \Lambda_\mu x^1$ in which we define $\omega_{\mu 1} = -\omega_{1\mu} \equiv \Lambda_\mu$.

The orbital part of transformation [10] is defined by

$$\begin{aligned} \delta_{orb} F_{MNP} &\equiv F_{MNP}(x_a + \omega_{ab}x^b) - F_{MNP}(x_a) \approx [\delta x_a] \cdot \partial^a F_{MNP} \\ &= [\omega_{ab}x^b \cdot \partial^a] F_{MNP} = [\omega_{1\mu}x^\mu \partial^1] F_{MNP} + [\omega_{\mu 1}x_1 \partial^\mu] F_{MNP} \\ &= [\Lambda_\mu x^\mu \partial^1] F_{MNP} - x^1 [\Lambda_\mu \partial^\mu] F_{MNP} \\ &= [(\Lambda \cdot x) \partial^1 - x^1 (\Lambda \cdot \partial)] F_{MNP} \equiv (\Lambda \cdot L) F_{MNP} \end{aligned} \quad (\text{A.2})$$

Note that δ_{orb} is independent of index MNP and is universal for all type tensor.

The spin part of transformation [10] becomes

$$\begin{aligned}
\delta_{spin} H_{\mu\nu\lambda} &\equiv \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{QRS}(x) - H_{\mu\nu\lambda} \\
&\approx \left[\frac{\partial(\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{1RS}(x) + \frac{\partial(\delta x^\sigma)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{\sigma RS}(x) \right] + \left[\frac{\partial x^Q}{\partial x^\mu} \frac{\partial(\delta x^1)}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{Q1S}(x) \right. \\
&\quad \left. + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial(\delta x^\sigma)}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{Q\sigma S}(x) \right] + \left[\frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial(\delta x^1)}{\partial x^\lambda} H_{QR1}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial(\delta x^\sigma)}{\partial x^\lambda} H_{QR\sigma}(x) \right] \\
&= \frac{\partial(\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{1RS}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial(\delta x^1)}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{Q1S}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial(\delta x^1)}{\partial x^\lambda} H_{QR1}(x) \\
&= \left[-\Lambda_\mu H_{1\nu\lambda} \right] + \left[-\Lambda_\nu H_{\mu 1\lambda} \right] + \left[-\Lambda_\lambda H_{\mu\nu 1} \right] \tag{A.3}
\end{aligned}$$

and $\delta H_{\mu\nu\lambda} = \delta_{orb} H_{\mu\nu\lambda} + \delta_{spin} H_{\mu\nu\lambda}$

In a same way

$$\begin{aligned}
\delta_{spin} H_{\mu\nu 1} &\equiv \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^1} H_{QRS}(x) - H_{\mu\nu 1} \\
&\approx \left[\frac{\partial(\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^1} H_{1RS}(x) + \frac{\partial(\delta x^\sigma)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^1} H_{\sigma RS}(x) \right] + \left[\frac{\partial x^Q}{\partial x^\mu} \frac{\partial(\delta x^1)}{\partial x^\nu} \frac{\partial x^S}{\partial x^1} H_{Q1S}(x) \right. \\
&\quad \left. + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial(\delta x^\sigma)}{\partial x^\nu} \frac{\partial x^S}{\partial x^1} H_{Q\sigma S}(x) \right] + \left[\frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial(\delta x^1)}{\partial x^1} H_{QR1}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial(\delta x^\lambda)}{\partial x^1} H_{QR\lambda}(x) \right] \\
&= \frac{\partial(\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^1} H_{1RS}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial(\delta x^1)}{\partial x^\nu} \frac{\partial x^S}{\partial x^1} H_{Q1S}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial(\delta x^\lambda)}{\partial x^1} H_{QR\lambda}(x) \\
&= 0 + 0 + \Lambda^\lambda H_{\mu\nu\lambda} \tag{A.4}
\end{aligned}$$

and $\delta H_{\mu\nu 1} = \delta_{orb} H_{\mu\nu 1} + \delta_{spin} H_{\mu\nu 1} = (\Lambda \cdot L) H_{\mu\nu 1} + \Lambda^\lambda H_{\mu\nu\lambda}$.

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